

GRAPHICAL REPRESENTATION OF SOME OF THE SIMPLER ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE.

BY E. T. LITTLEWOOD, M.A., B.Sc.

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1. A uniform function $w=f(z)$, where $z=x+iy$, may be completely represented in a model in the following manner:—

Modulus.—At each point of the (horizontal) z plane erect a vertical line of height h equal to $|w|$. The upper extremities of these lines will form a surface which will be called the “modular surface.”

Argument.—Construct a family of curves in the z plane such that the tangent to one of them at any point is parallel to the vector w corresponding to that point. It will be shown in § 5 that these curves are the “stream lines” of a liquid moving irrotationally in a certain manner in two dimensions. Hence they will be called the “stream lines” of the function w .

The idea of the “modular surface,” and the complete representation of a function by means of a model I believe to be original.

The modular surface and stream lines of an analytic function have certain simple and interesting relations, as will now be shown.

2. *Contour Lines and Lines of Greatest Slope.*—Let a system of contour lines and their orthogonal trajectories, the lines of greatest slope, be traced on the modular surface. The orthogonal projections of the contour lines on the z plane are evidently lines along which h , i.e., $|w|$, is constant, or “lines of constant modulus,” while the projections of the lines of greatest slope are their orthogonal trajectories.

It can easily be shown from elementary principles that the latter curves are also lines of “constant argument,” and, since $\arg w$ is constant, it is evident that—

$$\frac{dh}{|dz|} = \frac{|dw|}{|dz|} = |w'|.$$

Thus the *gradient of the line of greatest slope* vertically above any point represents the *modulus of the differential coefficient* at that point. It will

further be seen that the *argument of the differential coefficient* is the *angle at which the stream line is cut by the line of constant argument*.

3. *Divergence and Curvature of Stream Lines*.—Suppose the stream lines cut by a family of orthogonal trajectories, which we may call “cross lines.” Let ds and dc be the respective (scalar) elements of arc of the stream line and corresponding cross line at z . If $\arg w = \psi$, we have $w = he^{i\psi}$, whence—

$$w' = \frac{w}{h} \frac{dh}{dz} + iw \frac{d\psi}{dz} \dots\dots\dots(1)$$

(a) Let dz be taken along the stream line. Then—

therefore—
 $\arg dz = \arg w \text{ and } |dz| = ds,$
 $\frac{dz}{ds} = \frac{w}{h} \dots\dots\dots(2)$

Now w' is constant for all directions at z . Hence, combining (1) and (2)—

$$w' = \frac{dh}{ds} + ih \frac{d\psi}{ds} \dots\dots\dots(3)$$

(b) Let dz be taken along the cross line. Then—

$\arg dz = \arg (iw),$
 $|dz| = dc \text{ and } |iw| = h,$
therefore—
 $\frac{dz}{dc} = i \frac{w}{h} \dots\dots\dots(4)$

Combining (1) and (4)—

$$w' = -i \frac{dh}{dc} + h \frac{d\psi}{dc} \dots\dots\dots(5)$$

From (3) and (5)—

$$\frac{dh}{ds} + ih \frac{d\psi}{ds} = -i \frac{dh}{dc} + h \frac{d\psi}{dc}.$$

Equating real and imaginary parts—

$$\frac{1}{h} \frac{dh}{ds} = \frac{d\psi}{dc}, \text{ and } -\frac{1}{h} \frac{dh}{dc} = \frac{d\psi}{ds}.$$

We may call $\frac{d\psi}{dc}$, the *curvature of the cross line*, the “divergence” of the stream line at the point.

Thus, as we proceed along a stream line the gradient of the line on the modular surface, which lies vertically over it, is proportional to the divergence, while as we cross a stream line from concave to convex the gradient is proportional to the curvature of the stream line.

Hence a straight stream line indicates that the modulus is stationary as we cross it, and parallel stream lines indicate that the modulus is constant as we proceed along them.

4. *Stream Lines at Zero Points.*—If $f(z)$ has a zero at a , we may in general put $f(z) = (z-a)\phi(z)$, where $\phi(z)$ is finite. Then—

$$\arg f(z) - \arg(z-a) = \arg \phi(a) \text{ ultimately,}$$

and is approximately constant for a small circle round a .

Hence the stream lines make a constant angle with the radii drawn from the zero.

Infinite points will not be discussed generally, but special cases occur in z^{-1} and $\log z$, treated below.

5. *Determination of Stream Lines.*—One or more of the following methods may be used:—

(1) At a point of a stream line whose angle of contingence is ψ —

$$\psi = \arg w = \arg f(x+iy),$$

and it may be possible to plot ψ at a number of arbitrary points from the principle that—

$$\arg w_1 + \arg w_2 = \arg (w_1 w_2),$$

and to sketch the stream lines in an approximative manner.

(2) Solution of the differential equation—

$$\frac{dy}{dx} = \tan \arg f(x+iy).$$

(3) The method of conjugate functions,* which is the method most generally useful.

Let—

$$w = u + iv,$$

then—

$$\frac{dz}{w} = \frac{dx + idy}{u + iv} = \frac{u dx + v dy}{u^2 + v^2} + i \frac{u dy - v dx}{u^2 + v^2}.$$

Both the real and imaginary parts of the last expression can be shown to be complete differentials. Hence we may write $\frac{dz}{w} = dC + idS$, where C and S are real functions of x and y .

* *Vide* acknowledgment at end of paper.

Integrating, we have—

$$C + iS = \int \frac{dz}{w} = \text{a function of } (x + iy).$$

Thus C is the velocity potential and S the stream function of a liquid moving irrotationally in two dimensions. Moreover, since along a stream line $u dy - v dx = 0$, and along a cross line $u dx + v dy = 0$, we see that $S = \text{constant}$, and $C = \text{constant}$ are the respective equations of the stream lines, as defined in § 1, and the corresponding cross lines.

The following case includes several of those which follow :—

If—

$$w = z^n, \quad n = 1$$

$$\begin{aligned} C + iS &= \int \frac{dz}{z^n} = -\frac{z^{-n+1}}{n-1} \\ &= -\frac{\cos(n-1)\theta - i \sin(n-1)\theta}{(n-1)r^{n-1}}, \end{aligned}$$

where r, θ are the polar co-ordinates of z . Hence the stream lines are given by—

$$\frac{\sin(n-1)\theta}{r^{n-1}} = \text{constant}.$$

6. *Construction of Models.*—I have constructed models,* in which the modular surface is suggested by a wire framework, through which the stream lines drawn on the horizontal base are visible. The wires for the most part follow vertical sections and contour lines, which were previously traced on paper from the functional equation.

In all cases except $\log z$ the functions are symmetrical about the real axis, and only the portion on one side of this is represented. This allows the framework to be attached to one side of a board, one face of which represents the vertical plane through the real axis.

It is possible that plaster of Paris or plasticine might be found a better material than wire.

7. *Illustrated Examples.*—The general characters of the models are shown in the following diagrams, which are accompanied by brief notes.

In each case but the last the thick line in the upper half represents the section of the modular surface by a vertical plane through the real axis, while the lower half represents (horizontal) stream lines.

A numerical factor k is introduced in most cases to reduce the elevation suitably, while the stream lines are unaffected.

* These were exhibited at the meeting.

1. $w = k(z - a).$

No diagram shown, but modular surface a right circular cone, with axis vertical and vertex at a .

Stream lines, found from $C + iS = \int \frac{dz}{z - a}$, a family of straight lines radial at a .

2. (i.) $w = k(z^2 - a^2).$ (Fig. 1.)

Modular surface rises from two conical zeros, and tends to a paraboloid of revolution at infinity.

Stream lines, found from—

$$C + iS = \int \frac{dz}{z^2 - a^2},$$

a system of coaxial circles intersecting at zeros.

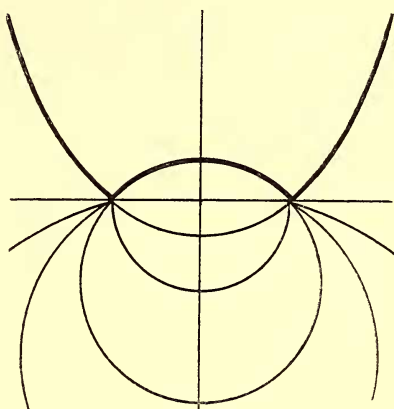


FIG. 1.

(ii.) $w = k(z^2 + a^2)$ (Fig. 2.)

Modular surface, the above turned through a right angle.

Stream lines, a system of coaxial circles, of which the zeros are limiting points.

By shifting the origin along the real axis the general quadratic function—

$$k(x^2 + px + q)$$

reduces to form (i.) or (ii.) according as $p^2 - 4q$ is positive or negative.

[In the models constructed, $k = .1$ and $a = 5$, unit of length being a centimetre.]

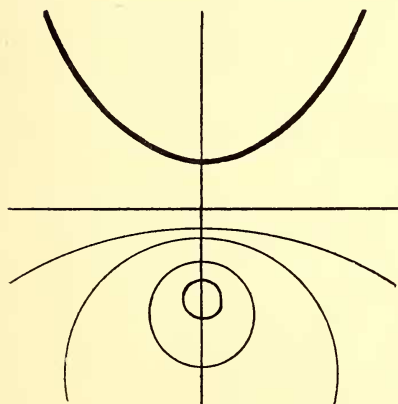


FIG. 2.

(iii.) $w = kz^2.$

Cases (i.) and (ii.) reduce to this when $a = 0$, and origin becomes a double zero. Modular surface a paraboloid of revolution. Stream lines a family of circles touching real axis at origin.

3.

$$w = \frac{1}{z} \quad (\text{Fig. 3.})$$

Modular surface obtained by rotating a rectangular hyperbola about an asymptote.

Stream lines given by—

$$r^2 \sin 2\theta = \text{constant},$$

a family of rectangular hyperbolas asymptotic to the real and imaginary axes. Origin an infinite point.

Note the infinite curvature at origin.

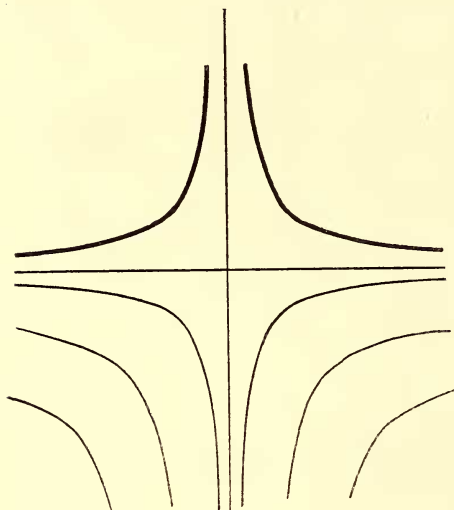


FIG. 3.

$$4. \quad (i.) \quad w = k(z+a)z(z-b) \quad (\text{Fig. 4}),$$

where a and b are real and positive.

[In the model constructed—

$$k = .02, \quad a = 5, \quad b = 10,$$

unit of length a centimetre.]

Modular surface rises from three zeros, and tends to a paraboloid of revolution of the third degree at infinity.

Stream lines might be found by method (3), but more easily plotted by method (1). They tend to Bernoulli's lemniscates at infinity, as in Fig. 6.

$$(ii.) \quad w = k(z^2 + a^2)(z - b). \quad (\text{Fig. 5.})$$

[In models constructed—

$$k = .01, \quad a = 4\sqrt{3}, \quad b = 12,$$

i.e., three zeros at angular points of equilateral triangle.]

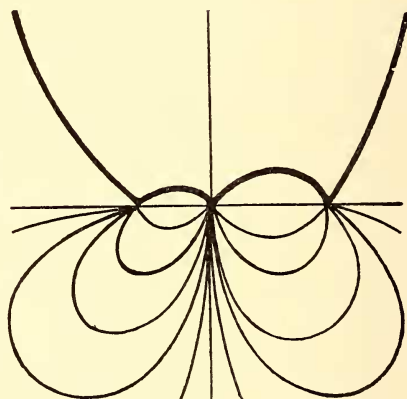


FIG. 4.

Stream lines plotted as in (i.). Also tend to Bernoulli's lemniscates at infinity, as in Fig. 6.

The above illustrate a cubic equation, (i.) with three real roots, (ii.) with two imaginary roots.

(iii.) $w = kz^3$. (Fig. 6.)

Cases (i.) and (ii.) reduce to this when—

$$a = b = 0.$$

Origin a triple zero.

Modular surface a paraboloid of revolution of third degree.

Stream lines, given by—

$$\frac{r^2}{\sin 2\theta} = \text{constant},$$

a family of Bernoulli's lemniscates.

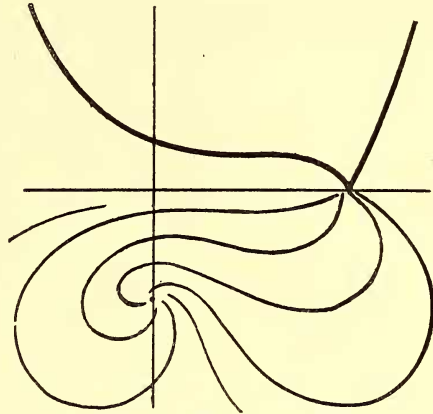


FIG. 5.

5. $w^2 = kz$. (Fig. 7.)

Modular surface obtained by rotating half a parabola about tangent at vertex.

Stream lines, given by $r(1 - \cos \theta) = \text{constant}$, a system of parabolas with origin as focus.

A case of a multiform function, the two sheets of the modular surface corresponding to the two branches being coincident. Note that one circuit round origin reverses direction of stream line, i.e., increases argument by π .

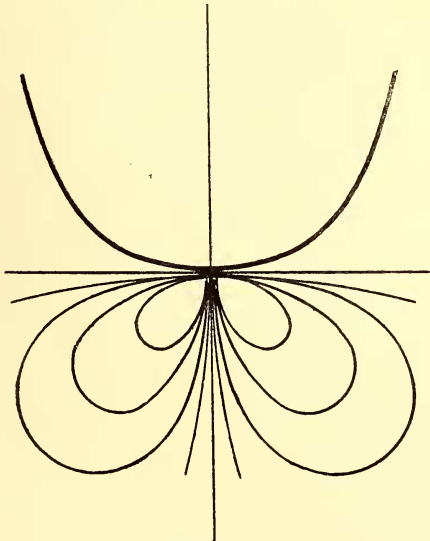


FIG. 6.

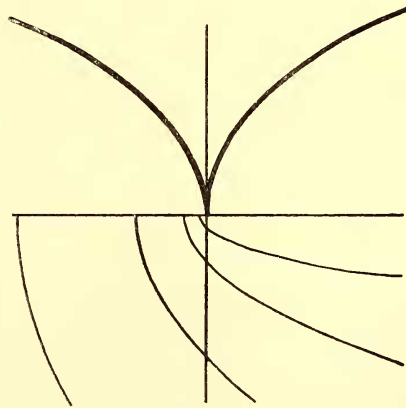


FIG. 7.

6. $w = e^z$. (Fig. 8.)

Modular surface cylindrical, with generators parallel to imaginary axis.

Vertical section the exponential curve.

Stream lines best found by method (2), by solving—

$$\frac{dy}{dx} = \tan y,$$

giving—

$$x + c = \log (\pm \sin y),$$

a system of curves asymptotic to—

$$y = \pm n\pi.$$

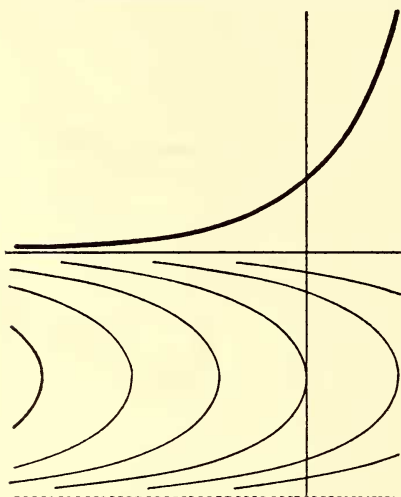


FIG. 8.

7. $w = \cos z$. (Fig. 9.)

[In model constructed, unit of length = 3 centimetres.]

Here—

$$h^2 = \frac{1}{2} (\cosh 2y + \cos 2x).$$

Modular surface rises from infinite row of zeros—

$$\pm \frac{1}{2}\pi, \pm \frac{3}{2}\pi, \&c.,$$

and rapidly tends to become cylindrical, with generators parallel to real axis.

Vertical section through real axis gives $\cos x$, with lower parts inverted, and is shown in upper part of figure.

Vertical sections through imaginary axis and parallel to it at distance $\frac{1}{2}\pi$ give $\cosh y$ and $\sinh y$ respectively, and are shown by thick and dotted lines in lower part of figure.

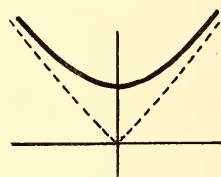
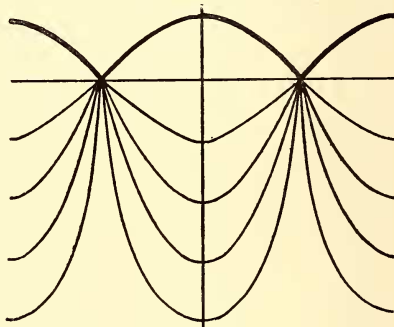


FIG. 9.

Stream lines, derived from—

$$C + iS = \int \frac{dz}{\cos z} \\ = \log (\sec z + \tan z),$$

given by—

$$\frac{\sinh y}{\cos x} = \text{constant},$$

a family of curves intersecting at zeros.

$$8. \quad w = \log z.$$

Here—

$$h^2 = (\log r)^2 + \theta^2.$$

Modular surface, a continuous spiral surface, winding upwards round a vertical axis through the origin.

When $r=1$, $h=\theta$, a helix of unit pitch. It can be shown to be a line of greatest slope, its projection on the z plane being a line of constant argument $\frac{1}{2}\pi$.

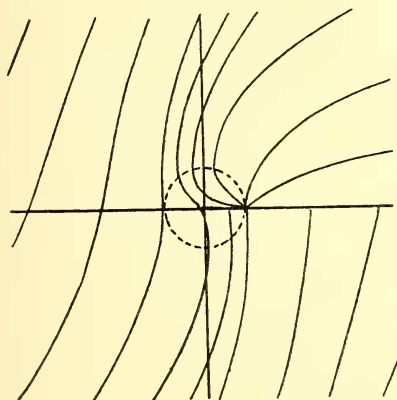


FIG. 11.

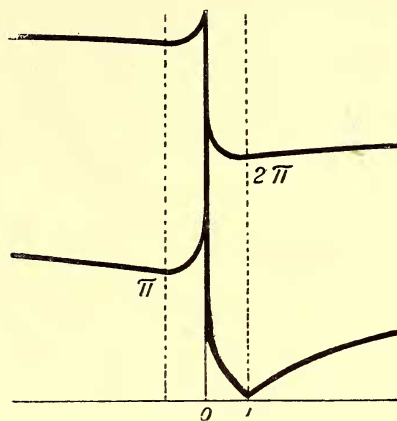


FIG. 10.

From this helix the surface curls up inwards round the axis and becomes asymptotic to it.

Vertical sections through real axis (Fig. 10), obtained when—

$$\theta = 0, \pi, 2\pi, \text{ \&c.}$$

Stream lines (Fig. 11) plotted from—

$$\tan \psi = \frac{\theta}{\log r}.$$

Discontinuity along real axis apparent only, as stream lines belong to different parts of surface.

[In the model the upper part is made wholly of wire tied by helical pieces and suitable vertical supports.]

In conclusion, I have to thank Mr. R. A. Herman, of Cambridge, for his criticism of the paper in its original form, as a consequence of which, while the results and diagrams are unaffected, I have improved the methods of arriving at them, in particular by adopting the method of conjugate functions in finding stream lines. I am also indebted to my son, Mr. J. E. Littlewood, for some useful suggestions, and to Professor Crawford, of the South African College, for criticism of the paper in its final form.